

# A Generalization of the Minisum and Minimax Voting Methods

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## Abstract

*In this paper, we propose a family of approval voting-schemes for electing committees based on the preferences of voters. In our schemes, we calculate the vector of distances of the possible committees from each of the ballots and, for a given  $p$ -norm, choose the one that minimizes the magnitude of the distance vector under that norm. The minisum and minimax methods suggested by previous authors and analyzed extensively in the literature naturally appear as special cases corresponding to  $p = 1$  and  $p = \infty$ , respectively. Supported by examples, we suggest that using a small value of  $p$ , such as 2 or 3, provides a good compromise between the minisum and minimax voting methods with regard to the weightage given to approvals and disapprovals. For large but finite  $p$ , our method reduces to finding the committee that covers the maximum number of voters, and this is far superior to the minimax method which is prone to ties. We also discuss extensions of our methods to ternary voting.*

## I. INTRODUCTION

In this paper, we consider the problem of selecting a committee of  $k$  members out of  $n$  candidates based on preferences expressed by  $m$  voters. The most common way of conducting this election is to allow each voter to select his favorite candidate and vote for him/her, and we select the  $k$  candidates with the most number of votes. While this system is easy to understand and implement, upon scrutiny, there arise certain unfavorable aspects.

Suppose we wish to elect a committee of size  $k = 1$ , and that there are two candidates, a Conservative  $A$ , and a Liberal  $B$ , with  $B$  expected to win.

In this case, it is to candidate  $A$ 's benefit to introduce a candidate  $C$  whose ideology is similar to that of candidate  $B$  in order to draw votes away from him, thereby ensuring his victory. This may be accomplished in practice by providing campaign funding, for example. This is illustrated by the example of the preferences of the voters indicated in Table 1. The first entry  $(A, B, C) = 30\%$  means that 30% of the voters rank the candidates in decreasing order of preference as  $A$  followed by  $B$ , and then by  $C$ . It is easy to see that as a result of these preferences, in an election between candidates  $A$  and  $B$  alone,  $B$  would win 56% of the votes. But in a three-way election,  $A$  would win, with the votes distributed as in Table 2. It is easy to see that similar problems can arise in the selection of committees with  $k > 1$  as well.

Preference vector	Percentage of votes
$(A, B, C)$	30
$(A, C, B)$	10
$(B, A, C)$	6
$(B, C, A)$	25
$(C, A, B)$	4
$(C, B, A)$	25

Table 1: Example of a ballot with full preference list indicated.

Candidate	Percentage of votes
$A$	40
$B$	31
$C$	29

Table 2: The ballot from Table 1 with only the top preference indicated.

A way to counteract this is to conduct preliminary elections within the party, in our case the Liberals, to select candidate  $B$  to represent them, and then conducting an election between candidates  $A$  and  $B$ .

The reason candidate  $B$  is preferable to  $A$  is that if we compare the two of them, a larger percentage of the population would prefer  $B$ . In fact, if we compare any two candidates,  $B$  is preferred. This type of candidate is called a *Condorcet winner*, and this criterion, that she be preferred by a majority in any pairwise comparison is called the *Condorcet criterion*.

To check whether the Condorcet criterion is satisfied, we may ask each voter to supply his/her full list of preferences, as in Table 1, and if there is a Condorcet winner, we can elect her to the committee. However, if there isn't a Condorcet winner, we do not yet have a method to choose the winner.

When we wish to elect a single winner (i.e.,  $k = 1$ ), we can use *Approval voting* [2]. We ask each voter to approve two candidates, and we select the candidate with the most number of approvals. In our example, assuming that

each voter approves of his top two preferences,  $A$  is approved by 50% of the voters,  $B$  is approved by 86%, and  $C$ , by 64%. This elects candidate  $B$ , as desired.

Approval voting is not a Condorcet method in general. A possible Condorcet method is to compare the candidates pairwise and choose the candidate who wins the most pairwise comparisons, and this is called the Copeland Method. However, if there is no Condorcet winner, this method will usually result in a tie.

In general, we would like to construct a voting rule which satisfies the following basic “fairness” criteria.

- If every voter prefers candidate  $A$  over  $B$ , the group also prefers candidate  $A$  over  $B$ .
- If every voter’s preference between  $A$  and  $B$  remains unchanged, then the group’s preference between  $A$  and  $B$  will also remain unchanged, even if voters’ preferences between other pairs change.
- Non-dictatorship: There is no single voter who possesses the power to determine the group’s preference.

While these conditions all seem reasonable, Arrow proved that there is no voting rule that satisfies all of them simultaneously [1, 8]. This is known as *Arrow’s Impossibility Theorem*.

In a realistic election, with, say, a million voters, and about twenty candidates, it is impractical to require the voters to provide a full list of preferences. Motivated by this problem, in this paper, we consider only approval voting and some of its variants.

## II. SHORTCOMINGS OF THE MINISUM AND MINIMAX VOTING METHODS

Suppose we have to elect a committee of  $k$  representatives from a pool of  $n$  candidates. Further, suppose there are  $m$  voters and each of them can approve of an arbitrary number of candidates.

We will consider the example used in [5] shown in Table 3. In this case  $n = 5$ ,  $m = 4$  and  $k = 2$ . We represent the ballot of voter 1 by the  $n$ -dimensional vector  $(1, 1, 0, 0, 1)$ , and similarly for all the other voters.

Now, we wish to select a committee of size  $k = 2$ , and the most common method is called the *minisum* method. This means we simply select the top  $k$  candidates with the most votes, in this case  $A$  and  $B$ . We represent this as the vector  $(1, 1, 0, 0, 0)$ . It is called the minisum strategy because, if you take the Hamming distances to each of the  $m$  ballots, in this case 1, 0, 2, 5, and add them, this is the vector that minimizes this sum. This result is easy to prove.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
1	1	1	0	0	1
2	1	1	0	0	0
3	1	1	1	1	0
4	0	0	1	1	1

Table 3: The example from [5] showing the approvals of the voters 1, 2 ... for the candidates *A*, *B* .... An approval is indicated by a “1”.

Let the candidates be numbered from 1 to  $n$ , and let the number of approvals for them be  $a_1, a_2, \dots a_n$  respectively. Without Loss Of Generality, assume

$$a_1 \geq a_2 \geq \dots \geq a_n.$$

Let the committee that minimizes the sum of Hamming distances be  $(i_1, i_2, \dots i_k)$ . This minimum value is equal to

$$(a_1 + a_2 + \dots + a_n) - (a_{i_1} + a_{i_2} + \dots a_{i_k}),$$

which, neglecting ties for the  $k^{\text{th}}$  position, is obviously minimized when

$$(i_1, i_2, \dots i_k) = (1, 2, \dots k).$$

This method does not weight disapprovals enough (as seen earlier in Table 2) and so, a method called the *minimax* was introduced [10]. This method chooses the committee which minimizes the maximum Hamming distance to the ballots. Under this, we elect the committee of *A* and *C*, i.e., the vector (1, 0, 1, 0, 0).

We will demonstrate the inadequacies of both of these methods by considering the example in Table 4, with  $m = 1000$  voters to elect a committee of  $k = 2$  members. We name the  $n = 4$  candidates  $A_1, A_2, B_1$  and  $B_2$  to indicate the correlation of ballots between candidates  $A_1$  and  $A_2$  and between  $B_1$  and  $B_2$ . For simplicity, in the rest of the paper, we restrict each of the voters to approve the same number of candidates as the size of the committee, in this case, two.

Candidate pair	Votes
$\{A_1, A_2\}$	500
$\{A_1, B_1\}$	100
$\{A_1, B_2\}$	10
$\{A_2, B_1\}$	20
$\{A_2, B_2\}$	20
$\{B_1, B_2\}$	350

Table 4: An example of a ballot which exhibits correlations of votes, possibly between candidates with the same ideology.

Using the minisum method, we would elect candidates  $A_1$  and  $A_2$ , as they have the most number of approvals, as shown in Table 5. However, this leaves 35% of the voters without either of their choices in the committee.

Candidate	Number of approvals
$A_1$	$500 + 100 + 10 = 610$
$A_2$	$500 + 20 + 20 = 540$
$B_1$	$100 + 20 + 350 = 470$
$B_2$	$10 + 20 + 350 = 380$

Table 5: The number of approvals per candidate, using the ballots from Table 4.

The minimax method is even worse because it does not narrow the choices at all; all of the  $\binom{n}{k} = \binom{4}{2} = 6$  committees yield the same minimax distance and it results in all ballots tied at a Hamming distance of 4.

Another problem with the minimax method is that it weights disapprovals too highly. A single voter can change the outcome of the entire election, which though it seems reasonable in the case of 4 voters, in more realistic scenarios with many more voters, it is not. Related papers [3, 5] do not elaborate upon this shortcoming but instead focus on efficient computation of the minimax solution.

An intuitively superior choice for the committee would be  $\{A_1, B_1\}$ , where most of the voters get at least one of their choices on the committee. We will demonstrate a method that leads to this result.

### III. THE $p$ -NORM MINIMIZATION METHOD

We represent each of the possible committees as a binary vector  $(c_1, c_2, \dots, c_n)$  with  $c_i = 1$  if candidate  $i$  is in the committee. Each of the  $m$  approval ballots is a vector of the same form,  $(b_1, b_2, \dots, b_n)$ . For each candidate committee-vector, we calculate the vector of Hamming distances of the ballot-vectors from it. This vector has size  $m$ , the number of voters. The components of this vector of Hamming distances are all even (This is a property of binary vectors of equal 1-norm.) In the case of the committee  $\{A_1, A_2\}$ , this vector will have a 1000 components, with 500 of these being 0, 150 of these being 2, and 350 of these being 4. For each of the candidate committee-vectors, this distance information is summarized in Table 6.

Suppose for a certain committee, the number of ballots at a Hamming distance of  $i$  is  $d_i$ . The  $p$ -norm of the Hamming distance vector for this committee is, therefore,

$$\left( \sum_i d_i \cdot i^p \right)^{1/p}.$$

We then find the committee that minimizes this  $p$ -norm.

We investigate the result of using intermediate values of  $p$  in Table 7. Minimizing the 1-norm yields the same solution as minisum; minimizing the  $\infty$ -norm yields the same solution as minimax. Notice that as the value of  $p \rightarrow \infty$ , the value of the distance tends to 4, the maximum Hamming distance.

At higher values of  $p$ , the number of voters who have not voted for a given committee is weighted higher, compared to the number of voters who have voted for at least one member of the committee. We suggest that using this method, choosing a small value of  $p$ , equal to 2 or 3, is a good compromise between the minisum and minimax solution, since it weights disapprovals more than minisum, but is much less likely to tie than minimax.

Note that for  $p$ , equal to 2 or 3, this method chooses the committee  $\{A_1, B_1\}$ , which contains at least one choice of 98% of the voters. This is the intuitively superior choice we identified earlier.

It is clear that this method generalizes for larger values of  $k$ , the committee size, with the distances ranging across all the even integers from 0 to  $2k$ .

### Maximum Coverage Problem ( $p \rightarrow \infty$ )

In this example, note that the optimal solution for  $p \geq 3$  always produces the committee  $\{A_2, B_1\}$ . Observe that this is the committee with the least number of voters who did not approve of any member in it, i.e., 1%. It is easy to show this is true in general: For sufficiently large but finite  $p$ , the  $p$ -norm method produces the solution with the least number of voters at distance  $2k$ , and therefore the committee with the “maximum coverage”.

Let the committees  $\mathcal{C}$  and  $\mathcal{C}'$  have  $d_i$  and  $d'_i$  voters respectively at a Hamming distance of  $i$ . Let the  $p$ -norm of the vector of Hamming distances from  $\mathcal{C}$  be  $\|\mathcal{C}\|_p$ . Since

$$\lim_{p \rightarrow \infty} \frac{\|\mathcal{C}\|_p}{\|\mathcal{C}'\|_p} = \lim_{p \rightarrow \infty} \frac{\left( \sum_{i=0}^{2k} d_i \cdot i^p \right)^{1/p}}{\left( \sum_{i=0}^{2k} d'_i \cdot i^p \right)^{1/p}} = \lim_{p \rightarrow \infty} \frac{\left( \sum_{i=0}^{2k} d_i \cdot \left( \frac{i}{2k} \right)^p \right)^{1/p}}{\left( \sum_{i=0}^{2k} d'_i \cdot \left( \frac{i}{2k} \right)^p \right)^{1/p}} = \left( \frac{d_{2k}}{d'_{2k}} \right)^{1/p},$$

Committee	# of ballots at distance 0	# of ballots at distance 2	# of ballots at distance 4
$\{A_1, A_2\}$	500	150	350
$\{A_1, B_1\}$	100	880	20
$\{A_1, B_2\}$	10	970	20
$\{A_2, B_1\}$	20	970	10
$\{A_2, B_2\}$	20	880	100
$\{B_1, B_2\}$	350	150	500

Table 6: Number of ballots at different distances from the possible committees, using the ballots from Table 4.

Committee	Value of $p$ in the $p$ -norm					
	1	2	3	4	10	100
$\{A_1, A_2\}$	<b>1700</b>	78.74	28.68	17.42	7.19	4.24
$\{A_1, B_1\}$	1840	<b>61.97</b>	<b>20.26</b>	11.77	5.42	4.12
$\{A_1, B_2\}$	2020	64.81	20.83	11.99	5.42	4.12
$\{A_2, B_1\}$	1980	63.56	20.33	<b>11.60</b>	<b>5.08</b>	<b>4.09</b>
$\{A_2, B_2\}$	2160	71.55	23.78	14.11	6.34	4.19
$\{B_1, B_2\}$	2300	92.74	32.14	19.00	7.45	4.26

Table 7:  $p$ -norm of the  $m$ -vector for the committees for different values of  $p$ , using the ballots from Table 4.

we see that for sufficiently large  $p$ , the committee with minimum  $p$ -norm is the one which has the minimum number of voters at distance  $2k$ . This committee minimizes the number of voters who disapprove of every one its members, and thereby maximizes the number of voters who approve of at least one of its members.

The maximum coverage problem [6] is a classic question in computer science and computational complexity theory, and in its standard form, it is phrased as follows:

You are given several overlapping sets and a number  $k$ . You must choose at most  $k$  of the sets such that the union of the selected sets is of maximum size.

The voting problem can be interpreted as an adaptation of this. The sets we are given are the voters who approved of a particular candidate. We are to choose  $k$  candidates in order to maximize the number of voters with at least one of their approved candidates on the committee. However, it is known that the maximum coverage problem is NP-hard. We will treat our example in Table 4 as an instance of this problem.

It is easy to see that the maximum cover in this case will be the committee  $\{A_2, B_1\}$ , covering 99% of the voters. This is the same result when  $p$  is large but finite. The greedy algorithm is not optimal in solving the maximum coverage problem, but applying it in the case of our example yields an interesting result. Applying the greedy algorithm, the first candidate we choose is  $A_1$ , since we can cover 61% of the voters with this. After choosing  $A_1$ , we can cover the most number of additional voters by choosing  $B_1$ , yielding the committee  $\{A_1, B_1\}$ , the same result as with  $p = 2$  and  $p = 3$ .

It is known that the greedy algorithm is a good approximation to the solution of the maximum coverage problem. [11]. Further, the greedy algorithm is computationally simple compared to the high computational complexity of calculating the Hamming distance vector for each of the many possible committees and minimizing the 2-norm. Therefore, the greedy algorithm applied to the problem of ensuring that the maximum number of voters get at least one

of their approved candidates on the committee is a computationally efficient, as well as reasonable, voting method.

### $p \rightarrow 0$

We now investigate the effect of letting  $p \rightarrow 0$ . In this case, it is the number of votes that the committee as a whole wins that matters, and not the number of votes of each of the constituent members. This can be shown using the same technique we used in the case of  $p \rightarrow \infty$ . Using the same notation as before, we get

$$\lim_{p \rightarrow 0} \frac{\|\mathcal{C}\|_p}{\|\mathcal{C}'\|_p} = \lim_{p \rightarrow 0} \frac{\left( \sum_{i=0}^{2k} d_i \cdot i^p \right)^{1/p}}{\left( \sum_{i=0}^{2k} d'_i \cdot i^p \right)^{1/p}} = \left( \frac{d_0}{d'_0} \right)^{1/p}.$$

The distinction will not be apparent in the previous example from Table 4, as result with  $p = 1$  coincides with the one with  $p = 0$ ; we will use a different example, shown in Table 8.

Candidate pair	Votes
$\{A_1, A_2\}$	300
$\{A_1, B_1\}$	250
$\{A_1, B_2\}$	150
$\{A_2, B_1\}$	40
$\{A_2, B_2\}$	60
$\{B_1, B_2\}$	200

Table 8: An example of a ballot to demonstrate the result of  $p \rightarrow 0$ .

We can see that the candidates with the most votes are  $A_1$  and  $B_1$ , as shown in Table 9, and according to the minisum method, i.e., with  $p = 1$ , the committee that will be elected is  $\{A_1, B_1\}$ . However, reducing the value of  $p$ , we see, from Table 10 that the elected committee changes to  $\{A_1, A_2\}$ , the pair that obtained the most votes, even though the candidate  $A_2$  received the fewest number of votes in total.

Candidate	Number of approvals
$A_1$	$300 + 250 + 150 = 700$
$A_2$	$300 + 40 + 60 = 400$
$B_1$	$250 + 40 + 200 = 490$
$B_2$	$150 + 60 + 200 = 410$

Table 9: The number of approvals per candidate, using the ballots from Table 8.



Committee	Value of $p$ in the $p$ -norm			
	1	0.5	0.1	0.001
$\{A_1, A_2\}$	1800	1107.11	<b>765.63</b>	<b>700.62</b>
$\{A_1, B_1\}$	<b>1620</b>	<b>1095.81</b>	808.45	750.56
$\{A_1, B_2\}$	1780	1225.51	914.08	850.62
$\{A_2, B_1\}$	2220	1445.51	1040.44	960.77
$\{A_2, B_2\}$	2380	1475.81	1026.70	940.83
$\{B_1, B_2\}$	2200	1307.11	880.50	800.76

Table 10:  $p$ -norm of the  $m$ -vector for the committees for different values of  $p$ , using the ballots from Table 8.

## IV. TERNARY VOTING

It has not escaped our notice that this method, of choosing the committee that minimizes the  $p$ -norm of vector of Hamming distances to the ballots, can be easily generalized to include ternary voting [7]. In this form of election, the voters choose a certain number of candidates to either approve (represented by a 1) or reject (represented with a  $-1$ ). Consider the simple example from Table 1 and let the voters be allowed 2 non-zero entries; the voters who have preference vector  $(A, B, C)$  can vote as  $(1, 1, 0)$ ,  $(1, 0, -1)$  or  $(0, -1, -1)$ .

In the case of only three representatives, the voters can actually indicate their full preference vector, but with more candidates, this is too complex to express. Moreover, it may be that the voter is actually neutral about most of the candidates, but rejects a few, and this form of ternary voting allows them to express this.

Under ternary voting, we need to replace the Hamming distance between a candidate committee,  $(c_1, c_2, \dots, c_n)$  and a ballot,  $(b_1, b_2, \dots, b_n)$  by some suitable metric. Since the Hamming metric corresponds to the  $L_1$ -metric, namely,  $\sum_{i=1}^n |c_i - b_i|$ , it is natural to consider the same metric under ternary voting.

However, we must revise our vector representation of the committees. If we represent the hypothetical committee consisting of the first two out of four candidates as  $(1, 1, 0, 0)$ , then the distance to the ballot  $(1, 1, 0, 0)$  will be 0, whereas the distance to the ballot  $(0, 0, -1, -1)$  will be 4. This is a bad model since a voter disapproving of the last two candidates is implicitly approving the first two. Therefore, this candidate committee is better represented by  $(1, 1, -1, -1)$  in which case the distance to both the ballots will be 2, in accordance with our intuition.

It is interesting to ask what the properties of the  $p$ -norm committees are, for various values of  $p$ , under ternary voting. Suppose candidate  $i$  has  $a_i$  approvals,  $r_i$  rejects, and  $n_i = m - a_i - r_i$  neutral votes. Then, candidate  $i$  contributes

$$n_i + 2r_i = m - a_i - r_i + 2r_i = m - (a_i - r_i)$$

to the 1-norm if he is in the committee, and

$$n_i + 2a_i = m - a_i - r_i + 2a_i = m + (a_i - r_i)$$

if he is not in the committee. Therefore, the minimum 1-norm committee consists of the  $k$  candidates with the  $k$  least values of  $m - (a_i - r_i)$ , or equivalently, the  $k$  highest values of  $a_i - r_i$ . This is an intuitively pleasing extension of the minisum method to ternary voting: the 1-norm committee minimizes the sum of the *net* approvals of its members, when rejects (explicit disapprovals) are allowed.

Similarly, we can extend our result of maximum coverage to ternary voting as well. Suppose we wish to elect a committee of  $k$  members from a set of  $n$ . Let each voter choose  $k$  candidates to either approve or reject. We see that the  $L_1$ -distance from the committees to the ballots ranges from  $n - k$  to  $n + k$ , and using the same method as before, we see that as  $p \rightarrow \infty$ , the committee that is selected is the one with the least number of ballots as distance  $n + k$ . It is clear that a ballot is only at this distance when none of the candidates that are approved are on the committee, *and* all the candidates that are rejected *are* on the committee. Therefore, in the context of the Maximum Coverage Problem, we consider a voter to be covered if any of the candidates he approves is on the committee, *or* if any of the candidates he rejects is *not* on the committee. Then, in this case also, choosing the committee that covers the maximum number of voters is better than using the minimax method because of the preponderance of ties that arise using the latter method.

## V. SUMMARY

We see that all three known methods—the committee with the most approvals, the minisum committee and the minimax committee—are all special cases of our  $p$ -norm method for  $p = 0$ ,  $p = 1$  and  $p = \infty$  respectively. However, using an intermediate value of  $p$  such as 2 or 3 is probably a better choice than any of these methods as a compromise between approvals and disapprovals under the Approval Voting method.

The minimax method is particularly unsuitable for this type of election given the preponderance of ties and should be logically replaced by the Maximum Cover method which has a well-known greedy algorithm approximation as well.

We also considered the application of our method to the problem of ternary voting and the meaning of the norm when  $p = 1$  and  $p \rightarrow \infty$  in this context. We see that our results easily extend to this case.

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